STAT 462/662
Introduction to Stochastic Processes

Homework 2

Due: Tuesday September 14, 2021

1. A Markov chain $X_0, X_1, \ldots$ on states 0, 1, 2 has the transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.5 & 0.3 \\ 0.5 & 0 & 0.5 \end{pmatrix}.$$

   Determine the conditional probabilities $P(X_2 = 1, X_3 = 1|X_1 = 0)$ and $P(X_1 = 1, X_2 = 1|X_0 = 0)$.

2. A Markov chain $X_0, X_1, \ldots$ on states 0, 1, 2 has the transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{pmatrix}.$$

   If it is known that the process starts in state 2 (that is, $X_0 = 2$), determine the probability $P(X_0 = 2, X_1 = 0, X_2 = 2)$.

3. A Markov chain $X_0, X_1, \ldots$ on states 0, 1, 2 has the transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{pmatrix}$$

   and initial distribution $P(X_0 = 0) = 0.2$, $P(X_0 = 1) = 0.7$ and $P(X_0 = 2) = 0.1$.
   Determine the probabilities $P(X_0 = 1, X_1 = 1, X_2 = 0)$ and $P(X_1 = 1, X_2 = 1, X_3 = 0)$.

4. Consider a gambler who at each play of the game has probability p of winning \$1 and probability $q = 1 - p$ of losing \$1. Assume that successive plays of the game are independent. The gambler's objective is to reach a total fortune of $\$N$, without first getting ruined (running out of money).

   Let X_n denote the player's fortune at time n. Then the process $\{X_n : n = 0, 1, \ldots\}$ is a Markov chain with transition probabilities

$$\begin{aligned} P_{00} &= P_{NN} = 1 \\ P_{i,i+1} &= p \\ P_{i,i-1} &= 1 - p = q \qquad \text{for } \; i = 1, 2, \ldots, N-1 \end{aligned}$$

   Let P_i ($i = 0, \ldots, N$) denote the probability that, starting with i, the gambler's fortune will *eventually* reach N.

(a) Show that

$$P_i = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 1/2 \\ \dfrac{i}{N} & \text{if } p = 1/2 \end{cases}$$

(b) Part (a) implies that, as $N \to \infty$,

$$\lim_{N\to\infty} P_i = \begin{cases} 1-(q/p)^i & \text{if } p > 1/2 \\ 0 & \text{if } p \leq 1/2 \end{cases}$$

Suppose that the gambler begins with \$100 and bets \$1 on each play, and it is also known that $p = 0.51$. When the gambler plays with an infinitely rich casino, what is the probability that the gambler's fortune will increase indefinitely?

5. An urn contains eight tags, of which four are red and four are green. Two tags are selected from the urn. If one tag is red and the other is green, then the selected tags are discarded and two blue tags are returned to the urn. Otherwise, the selected tags are resumed to the urn. This process repeats until the urn contains only blue tags. Let X_n denote the number of red tags in the urn after the nth draw, with $X_0 = 4$. (This is an elementary model of a chemical reaction in which red and green atoms combine to form a blue molecule.) Find the transition probability matrix.

   Hint: You may use a hypergeometric distribution to calculate a transition probability.

6. **(Stat 662 only; Bonus for 462):** The random variables $\xi_1, \xi_2, \ldots$ are independent and with the common probability mass function:

   $$P(\xi_i = 0) = 0.2, P(\xi_i = 1) = 0.3, P(\xi_i = 2) = 0.1, P(\xi_i = 3) = 0.4 \text{ for } i = 1, 2, \cdots.$$

   Let $X_0 = 0$, and let $X_n = \max\{\xi_1, \ldots, \xi_n\}$ be the largest ξ_i observed to date. Determine the transition probability matrix for the Markov chain $\{X_n\}$.