Discrete-time Markov chains (DTMC)

Recall: 1-step transition matrix

$$P_{ij} = P(X_{n+1} = j \mid X_n = i)$$

$$\mathbb{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ P_{20} & P_{21} & P_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

n-step transition probabilities

$$P_{ij}^{n} = P(X_{n+k} = j \mid X_k = i), \quad n \geq 0, \quad i, j, k \geq 0$$

[see next page of notes!]

Q. What is $P(X_n = j)$?

unconditional probability that after n steps the process will be in state j

To compute this, we need a vector of initial probabilities

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$$

where $\alpha_i = P(X_0 = i)$ for $i = 0, 1, 2, \ldots$

$$\sum_{i=0}^{\infty} \alpha_i = 1$$

$$P(X_n = j) = \sum_{i=0}^{\infty} \alpha_i P_{ij}^n = \alpha \cdot P_j^n$$

vector $\quad$ j^{th} column of P^n

More generally,

$$\alpha \cdot P^n = \alpha^n = (\alpha_0^n, \alpha_1^n, \alpha_2^n, \ldots)$$

where $\alpha_i^n = P(X_n = i)$ and these probabilities sum to 1

<u>Code this in R:</u>

$P^2 = P$ %*% P

$\alpha P_j^2 = \alpha$ %*% $P^2[\,,j]$ OR $\alpha P^2 = \alpha$ %*% P^2

$P^5 = P$ %^% 5

requires package "expm"

Do in Lec 10

Lec 9 9/26

②

$$\boxed{P_{ij}^{(n+m)} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{m}}$$

$\Leftarrow$ Chapman-Kolmogorov Equation

method for computing n-step transition probabilities

$$\Rightarrow P^{(n+m)} = P^{n} \cdot P^{m}$$

(Notation $P^{(k)} = P^{k}$
k-step transition matrix)

$\vdots$ details in book

$$P^{n} = \underbrace{P \cdot P \cdots P}_{n \text{ times}}$$

— multiply 1-step transition matrix by itself n times

$$P^{n} = \begin{bmatrix} P_{00}^{n} & P_{01}^{n} & \cdots & P_{0j}^{n} & \cdots \\ P_{10}^{n} & P_{11}^{n} & \cdots & P_{ij}^{n} & \cdots \\ \vdots & \vdots & & \vdots & \end{bmatrix}$$

j^{th} column : conditional probabilities that the process will be in state j after n steps

Classification of states

def: State j is accessible from state i if

$$P_{ij}^{n} > 0 \quad \text{for some } n \geq 0$$

i.e. Starting from state i, it is possible that the process will enter state j.

Example: Simple random walk on $\mathbb{Z}$ $(p = \frac{1}{2})$

j is accessible from i for any pair of states i & j

def: State j communicates with state i if

j is accessible from i & i is accessible from j.

Denote by $j \leftrightarrow i$.

Relation on the set of states:

since $P_{ii}^{0} = P(X_0 = i \mid X_0 = i) = 1$

- $i \leftrightarrow i \quad \forall i$ (i communicates with itself) "reflexive"
- If $i \leftrightarrow j$, then $j \leftrightarrow i$ (symmetric)
- If $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$ (transitive)

Note: This relation is an equivalence relation.

Divides the states into disjoint equivalence classes.

If $i \leftrightarrow j$, then i & j are in the same class.

State space consists of 1 or more classes.

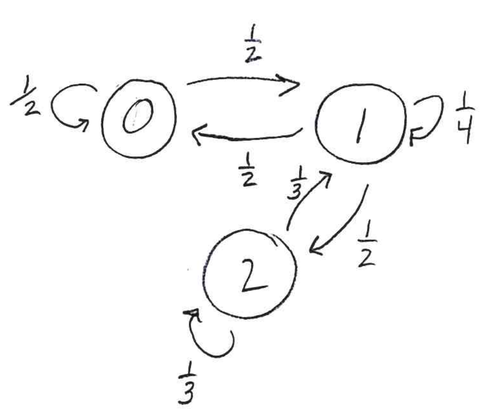
All states in the same class are accessible to one another.

def: If there is only 1 class (i.e. all states communicate with each other), then the Markov chain is irreducible.

Example 1: State Space $= \{0, 1, 2\}$

Transition Matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{array}$$



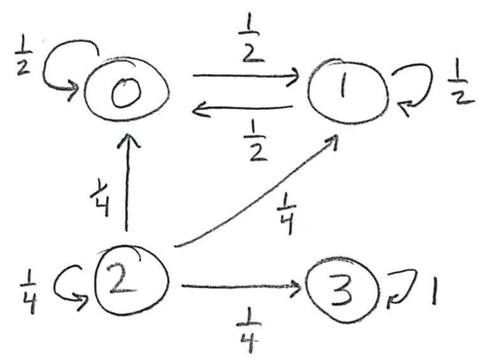
This chain is irreducible.

*R

Example 2: State space $= \{0, 1, 2, 3\}$

Transition Matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \!\!\! \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



Classes: $\{0,1\}$, $\{2\}$, $\{3\}$

why? $0 \leftrightarrow 1$ $\quad$ $2 \to 0$ $\quad$ $3 \leftrightarrow 3$

$\begin{pmatrix} 0 \leftrightarrow 0 \\ 1 \leftrightarrow 1 \end{pmatrix}$ $\quad$ $2 \to 1$

$2 \to 2$

$2 \to 3$

This chain is NOT irreducible.

State 3 is an absorbing state. If an absorbing state exists, the MC is not irreducible.

[Coming back to this example,

$\{0,1\}$ & $\{3\}$ recurrent classes

$\{2\}$ transient class

steady state dist'n is NOT unique!]